

Induced Gauge Structure of Quantum Mechanics on S^D

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(Received December 16,1996)

The Ohnuki-Kitakado (O-K) scheme of quantum mechanics on S^D embedded in R^{D+1} is investigated. Generators satisfying the O-K algebra are written down explicitly in term of the induced gauge potential. A direct method is developed to obtain the generators in covariant form. It is seen that there exists an induced gauge configuration which is trivial on S^D but might cause a nontrivial physical effect in R^{D+1} . The relation of the O-K scheme to extended objects such as the 't Hooft-Polyakov monopole is discussed.

§1. Introduction

Recently several authors have discussed the manner in which quantum mechanics should be formulated on manifolds.^{1)–3)} It turns out that, on a manifold, there exist some inequivalent quantization schemes, which can be identified with superselection sectors of the system. It is quite interesting that they could also discuss the origin of spin and gauge structure. Landsman and Linden,³⁾ developing the canonical group quantization of Isham,²⁾ discussed quantum mechanics on coset spaces. Regarding a circle S^1 and a sphere S^2 as R/Z and $SO(3)/SO(2)$, respectively, they found that the Aharonov-Bohm θ -angle and the Dirac magnetic monopole appear naturally in their quantum mechanics. Subsequently and independently, Ohnuki and Kitakado (O-K) investigated quantum mechanics on S^D embedded in R^{D+1} .^{4),5)} They found that the same gauge structures as those of Ref.3) emerge in the representation of their fundamental algebra of observables on S^D . They also observed that, in an appropriate limit, their quantum mechanics on S^D reduces to the usual quantum mechanics on R^D with the spin automatically built into the theory. On the other hand, McMullin and Tsutsui developed a generalized version of Dirac's quantization of a constrained system.^{6),7)} Regarding S^4 as $\text{Spin}(5)/\text{Spin}(4)$, they found that their H -connection reproduces a background BPST instanton and that the relativistic spin structure naturally arises in their quantization scheme. It should be noted that the induced gauge fields found in Ref.5) is in fact the H -connection.⁸⁾

In this paper, we follow the line of thought of O-K. Our discussion is made on the basis of their fundamental algebraic relations of observables. We develop a method which does not rely on Wigner's method of the little group. Discussion becomes simpler, in our opinion, and we directly obtain results in covariant form. We obtain a formula to express the O-K generator explicitly in term of the gauge potential and field strength. A gauge fixing condition leading us to the Wu-Yang Ansatz^{9),10)} for magnetic monopoles and the Belavin-Polyakov-Schwartz-Tyupkin (BPST) ansatz¹¹⁾ for instantons is given. Our discussion reveals that there exist three classes of solutions of the O-K algebra for any $D \geq 2$. Two of these three classes of solutions yield vanishing field strength on S^D and should be considered as trivial configurations on S^D . We discuss, however, that one of them might cause a nontrivial physical effect in R^{D+1} because of its singularity at the

origin of R^{D+1} . We also consider how O-K's quantum mechanics on S^D can include the physically allowed gauge configurations such as the 't Hooft-Polyakov monopole,^(12),13) Prasad-Sommerfield monopole,⁽¹⁴⁾ etc. We find that, if we wish to include the extended objects mentioned above, the radius of S^D should be taken much larger than the size of the object concerned.

This paper is organized as follows. In §2, we consider the general structure of the operator introduced by O-K. In §3, we impose a gauge condition and find the solutions of the O-K algebra. In §§4 and 5, we discuss the cases $D = 2$ and $D = 3$, respectively. The final section, §6, is devoted to summary.

§2. Structure of $G_{\alpha\beta}$

The D -dimensional sphere S^D embedded in R^{D+1} is defined by $x_\alpha \in R, \alpha = 1, 2, \dots, D+1$, satisfying

$$\sum_{\alpha=1}^{D+1} (x_\alpha)^2 = r^2, \quad (2 \cdot 1)$$

$$[x_\alpha, x_\beta] = 0, \quad (2 \cdot 2)$$

where r is a positive constant. O-K⁽⁵⁾ postulated that the fundamental algebra of quantum mechanics on $S^D \subset R^{D+1}$ is given by

$$[x_\lambda, G_{\alpha\beta}] = i(x_\alpha \delta_{\lambda\beta} - x_\beta \delta_{\lambda\alpha}), \quad (2 \cdot 3)$$

$$[G_{\alpha\beta}, G_{\lambda\mu}] = i(\delta_{\alpha\lambda} G_{\beta\mu} - \delta_{\alpha\mu} G_{\beta\lambda} + \delta_{\beta\mu} G_{\alpha\lambda} - \delta_{\beta\lambda} G_{\alpha\mu}), \quad (2 \cdot 4)$$

where $G_{\alpha\beta}, \alpha, \beta = 1, 2,$

$\dots, D+1$, are self-adjoint operators. They sought the irreducible unitary representations of $G_{\alpha\beta}$ with the aid of Wigner's technique to obtain the representation of the Poincaré

group. Expressing $G_{\alpha\beta}$ as

$$G_{\alpha\beta} = L_{\alpha\beta} + f_{\alpha\beta}(x) = -G_{\alpha\beta}, \quad (2 \cdot 5)$$

$$L_{\alpha\beta} = \frac{1}{i}(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \quad (2 \cdot 6)$$

O-K introduced the gauge potential $A_\alpha(x)$ by

$$r^2 A_\alpha(x) = \sum_{\beta=1}^{D+1} f_{\alpha\beta} x_\beta. \quad (2 \cdot 7)$$

From (2 · 7) and $f_{\alpha\beta}(x) = -f_{\beta\alpha}(x)$, we have

$$\sum_{\alpha=1}^{D+1} x_\alpha A_\alpha(x) = 0. \quad (2 \cdot 8)$$

The gauge transformation on S^D is caused by a unitary matrix $U(x)$ which is a representation of $SO(D+1)$ and satisfies

$$\mathcal{D}U(x) = 0. \quad (2 \cdot 9)$$

Here \mathcal{D} is the dilation operator defined by

$$\mathcal{D} = \sum_{\alpha=1}^{D+1} x_\alpha \partial_\alpha. \quad (2 \cdot 10)$$

The condition (2 · 8) is preserved under such a gauge transformation: $\sum_{\alpha=1}^{D+1} x_\alpha A_\alpha^U(x) = 0$, $A_\alpha^U(x) = U(x)A_\alpha(x)U^\dagger(x) + iU(x)\partial_\alpha U^\dagger(x)$.

To see what kind of gauge potential is allowed in the above scheme, we first obtain a formula expressing $f_{\alpha\beta}(x)$ by $A_\alpha(x)$. Although Eq. (2 · 7) cannot be solved algebraically w.r.t. $f_{\alpha\beta}(x)$, it is possible to obtain the desired formula in the following way. Substituting (2 · 5) into (2 · 4), we are led to

$$P_{\alpha\beta,\lambda\mu}[f] = i(x_\alpha \partial_\beta f_{\lambda\mu} - x_\beta \partial_\alpha f_{\lambda\mu}) - i(x_\lambda \partial_\mu f_{\alpha\beta} - x_\mu \partial_\lambda f_{\alpha\beta}), \quad (2 \cdot 11)$$

where $P_{\alpha\beta,\lambda\mu}[f]$ is defined by

$$P_{\alpha\beta,\lambda\mu}[f] = [f_{\alpha\beta}, f_{\lambda\mu}] - i(\delta_{\alpha\lambda} f_{\beta\mu} - \delta_{\alpha\mu} f_{\beta\lambda} + \delta_{\beta\mu} f_{\alpha\lambda} - \delta_{\beta\lambda} f_{\alpha\mu}). \quad (2 \cdot 12)$$

Multiplying (2 · 12) by $x_\mu x_\beta$, summing over μ and β , making use of (2 · 7) and (2 · 8), and noting the relations $\sum_{\mu=1}^{D+1} (\mathcal{D}f_{\lambda\mu})x_\mu = \mathcal{D}(r^2 A_\lambda) - r^2 A_\lambda$ and $\sum_{\mu=1}^{D+1} (\partial_\alpha f_{\lambda\mu})x_\mu = \partial_\alpha(r^2 A_\lambda) - f_{\lambda\alpha}$, we obtain

$$r^4[A_\alpha, A_\lambda] + ir^2 f_{\alpha\lambda} = i\{x_\alpha \mathcal{D}(r^2 A_\lambda) - x_\lambda \mathcal{D}(r^2 A_\alpha) - r^2 \partial_\alpha(r^2 A_\lambda) + r^2 \partial_\lambda(r^2 A_\alpha)\}. \quad (2 \cdot 13)$$

We see that the condition (2 · 8) yields

$$A_\alpha + \mathcal{D}A_\alpha = - \sum_{\beta=1}^{D+1} F_{\alpha\beta} x_\beta, \quad (2 \cdot 14)$$

where $F_{\alpha\beta}(x)$ is the field strength defined by

$$F_{\alpha\beta}(x) = i[D_\alpha, D_\beta] = \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x) - i[A_\alpha(x), A_\beta(x)], \quad (2 \cdot 15)$$

with D_α being the covariant derivative

$$D_\alpha = \partial_\alpha - iA_\alpha(x). \quad (2 \cdot 16)$$

We now obtain from (2 · 13), (2 · 14) and (2 · 15) that

$$f_{\alpha\beta}(x) = -\{x_\alpha A_\beta(x) - x_\beta A_\alpha(x)\} + H_{\alpha\beta}(x), \quad (2 \cdot 17)$$

$$H_{\alpha\beta}(x) = - \sum_{\gamma=1}^{D+1} x_\gamma \{F_{\alpha\beta}(x)x_\gamma + F_{\beta\gamma}(x)x_\alpha + F_{\gamma\alpha}(x)x_\beta\}. \quad (2 \cdot 18)$$

Equations (2 · 5) and (2 · 17) lead us to

$$G_{\alpha\beta} = M_{\alpha\beta} + H_{\alpha\beta}(x), \quad (2 \cdot 19)$$

where $M_{\alpha\beta}$ is defined by

$$M_{\alpha\beta} = \frac{1}{i}(x_\alpha D_\beta - x_\beta D_\alpha). \quad (2 \cdot 20)$$

Since D_α and $F_{\alpha\beta}(x)$ are gauge covariant, the gauge covariance of $G_{\alpha\beta}$ is manifest in (2 · 19). We thus understand that the fundamental algebraic relations (2 · 1) ~ (2 · 4) are gauge invariant.

We note that $G_{\alpha\beta}$ can be expressed solely by $M_{\alpha\beta}$ as follows. By definitions (2 · 12) and (2 · 20), we have

$$P_{\alpha\beta,\lambda\mu}[M] = i(x_\alpha x_\lambda F_{\beta\mu} - x_\alpha x_\mu F_{\beta\lambda} + x_\beta x_\mu F_{\alpha\lambda} - x_\beta x_\lambda F_{\alpha\mu}). \quad (2 \cdot 21)$$

Putting $\beta = \mu$ in (2 · 21) and summing over μ , we obtain

$$\sum_{\mu=1}^{D+1} [M_{\alpha\mu}, M_{\lambda\mu}] - i(D-1)M_{\alpha\lambda} = -iH_{\alpha\lambda}(x). \quad (2 \cdot 22)$$

Equations (2 · 19) and (2 · 22) yield

$$G_{\alpha\beta} = DM_{\alpha\beta} + i \sum_{\mu=1}^{D+1} [M_{\alpha\mu}, M_{\beta\mu}]. \quad (2 \cdot 23)$$

§3. Solutions of the fundamental algebra

In this section, we seek solutions of the algebraic relations (2 · 1) ~ (2 · 4) . We denote the representation of the generators of the gauge group $SO(D + 1)$ by $S_{\alpha\beta} = -S_{\beta\alpha}$, $\alpha, \beta = 1, 2,$

cdots, $D + 1$. They satisfy

$$P_{\alpha\beta, \lambda\mu}[S] = 0 \quad (3 \cdot 1)$$

and can be normalized as

$$tr(S_{\alpha\beta}S_{\lambda\mu}) = \sigma(\delta_{\alpha\lambda}\delta_{\beta\mu} - \delta_{\alpha\mu}\delta_{\beta\lambda}), \quad (3 \cdot 2)$$

where σ is a positive constant independent of α, β, λ and μ . The gauge potential $A_\alpha(x)$ is written, without loss of generality, as

$$A_\alpha(x) = \sum_{\beta, \gamma, \delta=1}^{D+1} E_{\alpha\beta\gamma\delta}(x) x_\beta S_{\gamma\delta}, \quad (3 \cdot 3)$$

where $E_{\alpha\beta\gamma\delta}(x)$ is a function satisfying $E_{\alpha\beta\gamma\delta}(x) = -E_{\alpha\beta\delta\gamma}(x)$. To fix the transformation properties of $E_{\alpha\beta\gamma\delta}(x)$ under the coordinate transformation $x_\alpha \rightarrow x'_\alpha = \sum_{\beta=1}^{D+1} \Lambda_{\alpha\beta} x_\beta$, $\Lambda = (\Lambda_{\alpha\beta}) \in SO(D + 1)$, we must fix that of $S_{\alpha\beta}$. Here we require so that $S_{\alpha\beta}$ transforms as

$$S'_{\alpha\beta} = \sum_{\gamma, \delta=1}^{D+1} \Lambda_{\alpha\gamma} \Lambda_{\beta\delta} S_{\gamma\delta}. \quad (3 \cdot 4)$$

Then the vector property $A'_\alpha(x') \equiv \sum_{\beta, \gamma, \delta=1}^{D+1} E'_{\alpha\beta\gamma\delta}(x') x'_\beta S'_{\gamma\delta} = \sum_{\beta=1}^{D+1} \Lambda_{\alpha\beta} A_\beta(x)$ indicates that $E_{\alpha\beta\gamma\delta}(x)$ is a fourth-rank tensor: $E'_{\alpha\beta\gamma\delta}(x') = \sum_{\kappa, \rho, \lambda, \sigma=1}^{D+1} \Lambda_{\alpha\kappa} \Lambda_{\beta\rho} \Lambda_{\gamma\lambda} \Lambda_{\delta\sigma} E_{\kappa\rho\lambda\sigma}(x)$. The

structure of the tensor $E_{\alpha\beta\gamma\delta}(x)$ should be fixed by the condition (2·8) and a gauge fixing condition. We impose the following gauge fixing condition

$$\sum_{\beta,\gamma=1}^{D+1} [\{\text{tr}(S_{\alpha\beta}S_{\beta\gamma})\}A_{\gamma}(x) - S_{\alpha\beta}\text{tr}\{S_{\beta\gamma}A_{\gamma}(x)\}] = 0. \quad (3 \cdot 5)$$

We can argue that, for any $A_{\alpha}(x)$ satisfying (2·8), there exists a gauge transformation $U(x)$ satisfying (2·9) such that $A_{\alpha}^U(x)$ obeys (2·8) and (3·5). From (3·2), (3·3) and (3·5), we obtain

$$\begin{aligned} E_{\alpha\beta\gamma\delta}(x) &= \delta_{\alpha\gamma}e_{\beta\delta}(x) - \delta_{\alpha\delta}e_{\beta\gamma}(x), \\ e_{\beta\delta}(x) &= \frac{1}{D} \sum_{\alpha=1}^{D+1} E_{\alpha\beta\alpha\delta}(x). \end{aligned} \quad (3 \cdot 6)$$

The condition (2·8) then yields $e_{\beta\delta}(x) = J_{\beta}(x)x_{\delta}$, where $J_{\beta}(x)$ is a vector field. Putting the scalar $\sum_{\alpha=1}^{D+1} x_{\alpha}J_{\alpha}(x)$ as $\frac{1}{2}V(r)$, we are led to

$$\sum_{\beta=1}^{D+1} E_{\alpha\beta\gamma\delta}(x)x_{\beta} = -\frac{1}{2}V(r)(x_{\gamma}\delta_{\alpha\delta} - x_{\delta}\delta_{\alpha\gamma}), \quad (3 \cdot 7)$$

which is, for $D = 3$, equivalent to the ansatz adopted by BPST in their pioneering paper on the instanton.¹¹⁾ We stress that, in the present context associated with the condition (2·8), the BPST Ansatz corresponds to the gauge condition (3·5). From (3·3) and (3·7), we obtain

$$\begin{aligned} A_{\alpha}(x) &= \frac{1}{2} \sum_{\beta,\gamma=1}^{D+1} A_{\alpha}^{\beta\gamma}(x)S_{\beta\gamma}, \\ A_{\alpha}^{\beta\gamma}(x) &= -V(r)(x_{\beta}\delta_{\alpha\gamma} - x_{\gamma}\delta_{\alpha\beta}) \end{aligned} \quad (3 \cdot 8)$$

and hence

$$A_{\alpha}(x) = V(r) \sum_{\beta=1}^{D+1} S_{\alpha\beta}x_{\beta}. \quad (3 \cdot 9)$$

It is evident that, for $D = 2$, the expression (3·9) for $A_{\alpha}(x)$ coincides with that adopted by Wu and Yang.⁹⁾ The field strength is now given by

$$F_{\alpha\beta}(x) = h(r)S_{\alpha\beta} + j(r)J_{\alpha\beta}(x), \quad (3 \cdot 10)$$

where $h(r)$, $j(r)$ and $J_{\alpha\beta}(x)$ are defined by

$$h(r) = r^2V(r)^2 - 2V(r), \quad (3 \cdot 11)$$

$$j(r) = r^2 V(r)^2 + r V'(r), \quad (3 \cdot 12)$$

$$J_{\alpha\beta}(x) = \sum_{\gamma=1}^{D+1} (\hat{x}_\alpha S_{\beta\gamma} - \hat{x}_\beta S_{\alpha\gamma}) \hat{x}_\gamma, \quad \hat{x}_\gamma = \frac{x_\gamma}{r}. \quad (3 \cdot 13)$$

It is straightforward to obtain

$$G_{\alpha\beta} = L_{\alpha\beta} - r^2 h(r) S_{\alpha\beta} - r^2 \{V(r) + h(r)\} J_{\alpha\beta}, \quad (3 \cdot 14)$$

where we have made use of the identity

$$\sum_{\gamma=1}^{D+1} \{J_{\alpha\beta}(x) x_\gamma + J_{\beta\gamma}(x) x_\alpha + J_{\gamma\alpha}(x) x_\beta\} x_\gamma = 0. \quad (3 \cdot 15)$$

We see that $G_{\alpha\beta}$ is independent of the derivative of $V(r)$.

The commutation relations among $J_{\alpha\beta}(x)$, $S_{\alpha\beta}$ and $L_{\alpha\beta}$ are calculated to be

$$[J_{\alpha\beta}, J_{\gamma\delta}] = i K_{\alpha\beta, \gamma\delta} \quad (3 \cdot 16)$$

$$[S_{\alpha\beta}, J_{\gamma\delta}] = -i K_{\alpha\beta, \gamma\delta} - i N_{\alpha\beta, \gamma\delta} \quad (3 \cdot 17)$$

$$[L_{\alpha\beta}, J_{\gamma\delta}] = i K_{\alpha\beta, \gamma\delta} + i N_{\gamma\delta, \alpha\beta} \quad (3 \cdot 18)$$

where $K_{\alpha\beta, \gamma\delta}$ and $N_{\alpha\beta, \gamma\delta}$ are defined by

$$K_{\alpha\beta, \gamma\delta} = \hat{x}_\alpha \hat{x}_\gamma S_{\beta\delta} - \hat{x}_\beta \hat{x}_\gamma S_{\alpha\delta} + \hat{x}_\beta \hat{x}_\delta S_{\alpha\gamma} - \hat{x}_\alpha \hat{x}_\delta S_{\beta\gamma}, \quad (3 \cdot 19)$$

$$N_{\alpha\beta, \gamma\delta} = (\hat{x}_\gamma \delta_{\beta\delta} - \hat{x}_\delta \delta_{\beta\gamma}) \sum_{\kappa=1}^{D+1} S_{\alpha\kappa} \hat{x}_\kappa - (\hat{x}_\gamma \delta_{\alpha\delta} - \hat{x}_\delta \delta_{\alpha\gamma}) \sum_{\kappa=1}^{D+1} S_{\beta\kappa} \hat{x}_\kappa. \quad (3 \cdot 20)$$

We now obtain

$$\begin{aligned} & P_{\alpha\beta, \gamma\delta}[G] \\ &= \{(r^2 h)^2 + r^2 h\} [S_{\alpha\beta}, S_{\gamma\delta}] - i r^2 (h + V) (r^2 h - r^2 V + 2) K_{\alpha\beta, \gamma\delta} \\ &\quad - i r^4 (h + V) h (N_{\alpha\beta, \gamma\delta} - N_{\gamma\delta, \alpha\beta}) \\ &= r^2 V (r^2 V - 1) (r^2 V - 2) \{(r^2 V - 1) [S_{\alpha\beta}, S_{\gamma\delta}] \\ &\quad - i (r^2 V - 1) K_{\alpha\beta, \gamma\delta} - i r^2 V (N_{\alpha\beta, \gamma\delta} - N_{\gamma\delta, \alpha\beta})\}. \end{aligned} \quad (3 \cdot 21)$$

The requirement (2 · 4), i.e.,

$$P_{\alpha\beta, \gamma\delta}[G] = 0 \quad (3 \cdot 22)$$

yeilds the result

$$r^2V\{r^2V - 1\}\{r^2V - 2\} = 0. \quad (3 \cdot 23)$$

We thus obtain three solutions

$$(a) \quad r^2V(r) = 0, \quad (3 \cdot 24a)$$

$$(b) \quad r^2V(r) = 1, \quad (3 \cdot 24b)$$

$$(c) \quad r^2V(r) = 2, \quad (3 \cdot 24c)$$

and $G_{\alpha\beta}$ is given by

$$(a) \quad G_{\alpha\beta} = L_{\alpha\beta}, \quad (3 \cdot 25a)$$

$$(b) \quad G_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}, \quad (3 \cdot 25b)$$

$$(c) \quad G_{\alpha\beta} = L_{\alpha\beta} - 2J_{\alpha\beta}(x), \quad (3 \cdot 25c)$$

in the respective cases. It can be seen that both cases (a) and (c) yield vanishing field strengths on S^D . If we denote case (b) with the singlet representation for $S_{\alpha\beta}$ by (b_0) , the field strength for the case (b_0) is also vanishing on S^D . Regarding $(a) = (b_0)$ hereafter, cases (b_0) and (c) are trivial as the induced gauge potential on S^D . It seems that in the O-K approach based on Wigner's little group, the case (c) is absorbed into case (b_0) because they are gauge equivalent to each other. It should be noted, however, that the gauge configuration of (c) exhibits quite a different property from that of (b_0) at the origin of R^{D+1} . We shall discuss later the manner in which they differ.

§4. Magnetic monopole solution

Here we investigate the case $D = 2$. In contrast to the case $D \geq 3$, at least two of α, β, λ and μ in $P_{\alpha\beta, \lambda\mu}[G]$ coincide for $D = 2$. Because of the anti-symmetry $G_{\alpha\beta} = -G_{\beta\alpha}$, it is

sufficient to consider three cases in $P_{\alpha\beta,\lambda\mu}[G] = 0$: (i) $\alpha = \mu = 1, \beta = 2, \lambda = 3$, (ii) $\alpha = \mu = 2, \beta = 3, \lambda = 1$, (iii) $\alpha = \mu = 3, \beta = 1, \lambda = 2$. Equation (3 · 21) simplifies to

$$P_{\alpha\beta,\lambda\alpha}[G] = [H(x), \sum_{\kappa=1}^3 x_{\kappa} M_{\alpha\kappa}]$$

$$(\alpha\beta\lambda) = (123), (231), (312), \quad (4 \cdot 1)$$

where $H(x)$ is given by

$$H(x) = x_1 F_{23}(x) + x_2 F_{31}(x) + x_3 F_{12}(x). \quad (4 \cdot 2)$$

For the sake of comparison with earlier works, it is convenient to use

$$T_{\gamma} = \frac{1}{2} \sum_{\alpha,\beta=1}^3 \epsilon_{\gamma\alpha\beta} S_{\alpha\beta}, \quad (4 \cdot 3a)$$

$$[T_{\alpha}, T_{\beta}] = i \sum_{\gamma=1}^3 \epsilon_{\alpha\beta\gamma} T_{\gamma} \quad (4 \cdot 3b)$$

instead of $S_{\alpha\beta}$. Equation (3 · 8) then becomes

$$A_{\alpha}(x) = \sum_{\beta,\gamma=1}^3 \epsilon_{\alpha\beta\gamma} x_{\beta} T_{\gamma} V(r), \quad (4 \cdot 4)$$

which is nothing but the Wu-Yang ansatz⁹⁾ for a three-dimensional Yang-Mills field. The function $H(x)$ in (4 · 2) is calculated to be

$$H(x) = \{r^2 V(r)^2 - 2V(r)\} \left(\sum_{\gamma=1}^3 x_{\gamma} T_{\gamma} \right), \quad (4 \cdot 5)$$

and we find that the r.h.s. of (4 · 1) is given by

$$[H, \sum_{\gamma=1}^3 x_{\gamma} M_{\alpha\gamma}] = -ir^2 V(r) \{r^2 V(r) - 1\} \{r^2 V(r) - 2\} \{ \hat{x}_{\alpha} (\sum_{\gamma=1}^3 \hat{x}_{\gamma} T_{\gamma}) - T_{\alpha} \}. \quad (4 \cdot 6)$$

We find that the $V(r)^4$ -term in (3 · 21) cancels out in the $D = 2$ case. In the following, we discuss the three solutions of (3 · 22) given in (3 · 24) and (3 · 25).

Solution (a) is trivial and equivalent to (b₀) defined at the end of §3.

Solution (b) corresponds to the one obtained by O-K.⁵⁾ The gauge potential in this case is the Wu-Yang solution of the pure Yang-Mills field theory. This configuration is known to be gauge equivalent to the following:¹⁰⁾

$$A_\alpha(x) = \sum_{\gamma=1}^3 A_\alpha^\gamma(x) T_\gamma,$$

$$A_\alpha^1 = A_\alpha^2 = 0, \quad A_\alpha^3 = -\frac{1}{2r} \tan\left(\frac{1}{2}\theta\right) e_\phi, \quad (4 \cdot 7)$$

where (θ, ϕ) is the polar coordinate on S^2 , and e_ϕ is the unit vector in the direction of ϕ . As has been discussed by many people, this configuration describes a gauge potential caused by a point-like magnetic monopole.¹⁰⁾

We next consider the solution (c), which was not considered by O-K.⁵⁾ As we discussed in the last paragraph of §3, (c) is gauge equivalent to (b₀). Since the field strength vanishes for $r > 0$ in this case, the gauge potential can be expressed as a pure gauge in a simply connected domain which does not contain the origin: $A_\alpha = iU\partial_\alpha U^\dagger$. The operator $G_{\alpha\beta}$ is given by $G_{\alpha\beta} = UL_{\alpha\beta}U^\dagger$ and we can check (2·4) by $P_{\alpha\beta,\lambda\mu}[G] = UP_{\alpha\beta,\lambda\mu}[L]U^\dagger = 0$. Although any $G_{\alpha\beta}$ of the above form satisfies (2·4), we here obtain a highly specified form of $G_{\alpha\beta}$, (3·25c). This specification should be attributed to the gauge condition (3·5). We note that we can replace this condition by

$$\text{tr}\left\{\left(\sum_{\gamma=1}^3 x_\gamma T_\gamma\right) A_\alpha(x)\right\} = 0. \quad (4 \cdot 8)$$

Although this configuration does not correspond to the magnetic monopole, it is nontrivial in R^3 because of its singularity at the origin. The unitary matrix U for case (c) is given by $U = e^{i\pi S}$, $S = \hat{x}_1 T_1 + \hat{x}_2 T_2 + \hat{x}_3 T_3$. The structure of the singularity at the origin can be envisaged by calculating the quantity Q defined by

$$Q = \int_{R^3} \rho(x) d^3x = \int_{R^3} \sum_{i=1}^3 \partial_i \xi_i(x) d^3x, \quad (4 \cdot 9)$$

$$\rho(x) = i \sum_{i,j,k=1}^3 \epsilon_{ijk} \text{tr}(U A_i A_j A_k), \quad (4 \cdot 10)$$

$$\xi_i(x) = \sum_{j,k=1}^3 \epsilon_{ijk} \text{tr}(U A_j A_k). \quad (4 \cdot 11)$$

Although $\rho(x)$ vanishes for any $r > 0$, the r.h.s. of (4 · 9) is equal to $4i \int_{S^2} \text{tr}(e^{i\pi S} S) d\Omega$ and nonvanishing in general, implying that $\rho(x)$ has a δ -function singularity at the origin.

The field equation of the pure $SO(3)$ Yang-Mills theory under the Wu-Yang Ansatz (4 · 4) is given by¹⁰⁾

$$r^2 \frac{d^2}{dr^2} \{r^2 V(r)\} = r^2 V(r) \{r^2 V(r) - 1\} \{r^2 V(r) - 2\}. \quad (4 \cdot 12)$$

It is interesting to note that the algebraic requirement (2 · 4) reproduces all the solutions of (4 · 12) of the type $r^2 V(r) = \text{const.}$

We have obtained in the above the gauge configuration of a point-like monopole. On the other hand, we know some examples of extended monopoles, the 't Hooft-Polyakov^{12),13)} monopole, the Prasad-Sommerfield¹⁴⁾ monopole, etc., of the $SO(3)$ Yang-Mills-Higgs field theory. The gauge configurations corresponding to these examples still take the form of (4 · 4), but the function $V(r)$ in these cases does not satisfy the condition (3 · 23). We find, however, that the function $r^2 V(r) - 1$ for the 't Hooft-Polyakov as well as the Prasad-Sommerfield monopoles decreases exponentially for large values of r :

$$r^2 V(r) - 1 \approx \text{const.} r e^{-\beta r}, \quad (r \approx \infty) \quad (4 \cdot 13)$$

where β^{-1} is the size parameter. Thus, instead of $P_{\alpha\beta, \lambda\alpha}[G] = 0$ ($\infty > r > 0$), we have

$$P_{\alpha\beta, \lambda\alpha}[G] \approx \text{const.} r e^{-\beta r} \{ \hat{x}_\alpha \left(\sum_{\gamma=1}^3 \hat{x}_\gamma T_\gamma \right) - T_\alpha \}, \quad (r \approx \infty)$$

$$(\alpha\beta\lambda) = (123), (231), (312). \quad (4 \cdot 14)$$

In other words, the condition

$$P_{\alpha\beta, \lambda\alpha}[G] \approx 0, \quad (r \gg \beta^{-1} > 0) \quad (4 \cdot 15)$$

allows for gauge configurations of the extended monopole of the above type. It should be noted that the Higgs field is concerned with the dynamics of a particle on S^2 but not with its kinematics. Since the fundamental algebra should be independent of the dynamics, only the Yang-Mills field appeared in the above discussion. Of course, the details of the gauge configuration cannot be determined only through (4 · 15).

§5. BPST instanton solution

In this section, we consider the case $D = 3$, i.e., the O-K algebra for S^3 embedded in R^4 . Results for $V(r)$ and $G_{\alpha\beta}$ are given by (3 · 24) and (3 · 25). Case (a) is trivial and identical with (b₀), as mentioned in §3. Solution (b) was discussed by O-K.⁵⁾ For this solution, we see that $\text{tr}\{\sum_{\mu,\nu=1}^4 (F_{\mu\nu})^2\}$ is proportional to r^{-4} , and the corresponding action integral is divergent. Its singularity structure at the origin was discussed by O-K in detail.

On the other hand, we have

$$F_{\alpha\beta}(x) = 0, \quad (r > 0) \quad (5 \cdot 1)$$

for the solution (c). Comparing with BPST,¹¹⁾ however, this solution should be interpreted as the zero size limit of the BPST solution. To understand the above interpretation, we replace $V(r) = 2r^{-2}$ by $V_\lambda(r) = 2(r^2 + \lambda^2)^{-1}$, where λ is the size parameter which can be taken as small as desired. The field strength then becomes

$$F_{\alpha\beta}^\lambda(x) = -\frac{4\lambda^2}{(r^2 + \lambda^2)^2} S_{\alpha\beta}, \quad (5 \cdot 2)$$

which is the configuration considered by BPST.¹¹⁾ Another way of understanding the above interpretation is to calculate the $SU(2)$ instanton number, q , corresponding to the configuration $A_\alpha^{\beta\gamma}(x) = -2(x_\beta\delta_{\alpha\gamma} - x_\gamma\delta_{\alpha\beta})r^{-2}$. Faithfully following the method of BPST,¹¹⁾ we obtain $q = \pm 1$. Two values, $+1$ and -1 , for q are allowed because there are two ways to reduce the $SO(4)$ gauge potential to the $SU(2)$ gauge potential. We expect that this configuration might cause a nontrivial effect for physics in R^4 and the instanton number q plays a similar role to that of the thin magnetic flux in the Aharonov-Bohm effect.

We note here some differences between previous works and those presented in this paper. Fujii, Kitakado and Ohnuki¹⁵⁾ considered quantum mechanics on S^{2n} , $n = 2, 3$, *cdots*, embedded in R^{2n+1} . They stereographically projected their $2n + 1$ dimensional gauge potential to that of S^{2n} and obtained a generalized BPST configuration¹⁶⁾ with a nonvanishing scale parameter. On the contrary, we considered the O-K algebra for S^3 in

R^4 and obtained the BPST configuration in R^4 with a vanishing size parameter. We here encounter a situation similar to that of the previous section: the O-K algebra excludes extended objects. If we adopt (4 · 15) instead of (2 · 4), we are allowed to include the BPST configuration with finite size.

§6. Summary

We have investigated the representation of the Ohnuki-Kitakado algebra (2 · 1) ~ (2 · 4) for quantum mechanics on S^D embedded in R^{D+1} without using Wigner's little group method. The function $f_{\alpha\beta}(x)$ in (2 · 5) was represented by (2 · 17) in terms of $A_\alpha(x)$ and $F_{\alpha\beta}(x)$. The expressions (2 · 19) and (2 · 23) for $G_{\alpha\beta}$ manifestly exhibit its gauge covariance, implying the gauge invariance of the O-K algebra. We have observed that the gauge condition (3 · 5) naturally leads us to the Wu-Yang ansatz for magnetic monopoles and the BPST ansatz for pseudoparticle solutions. Three classes of solutions of the O-K algebra were obtained: (a), (b) and (c). Class (a) which is identical to the (b₀), the singlet representation case of (b), is trivial both on S^D and R^{D+1} . Class (b) is the one discussed by O-K and nontrivial on S^D . The field strength for the class (c) vanishes on S^D , implying that the configuration is trivial on S^D . Configurations belonging to this class, however, might produce some physical effects in R^{D+1} . It was noted for the case $D = 2, 3$ that the Q of (4 · 9) and the instanton number q might play the role similar to that of the thin magnetic flux in the Aharonov-Bohm effect. We have also discussed how the gauge configurations of extended objects such as the 't Hooft-Polyakov monopole can be included in the O-K scheme of quantum mechanics in S^D .

Acknowledgements

One of the authors (M.H.) is grateful to Professor Z. Z. Xin for his hospitality at Liaoning University, China, where a part of this paper was written.

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